GENERAL STORY

In AA I we studied groups, particular attention was paid to abelian vs non-abelian gps. We introduced

cosets → Normal subgps → Quotients → homomorphisms
↓ Lagrange
↓ structure ← (ie first hom. thm)

Normal subgps provide some measure of abelian-ness of non abelian groups.
In abelian gps <u>every</u> subgp is normal, and the fewer normal subgps a gp has the "less" abelian it is.
Recall quadratics: $ax^2+bx+c=0$ $\left(\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)$
Similar formula for deg 3 (Cardano) & 4 (Ferrari)
what about deg 5? The famous Abel theorem says: No sol^n for $ax^5+\dots=0$ in general.

No surprise Abel's name is here!
By the end of the course we will associate w/ polynomials w/ groups! In particular, symmetry groups governing the interactions of their roots. Whether the polynomials have sol'ns then depends on the abelian-ness of these groups! This is the celebrated theory of Galois.

SOLVABLE GROUPS

We will begin our study w/ solubility. This, informally, is a measure of how far a group is from being commutative. By the end of the semester we will connect this notion with "solvability" of polynomial equations.

COMMUTANT

Let us begin by recalling that a group G is called <u>commutative</u> if $ab = ba$ $\forall a, b \in G$.

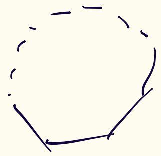
Rearranging, this is equivalent to $aba^{-1}b^{-1} = e$.

Defn: The subcollection of elements $\{aba^{-1}b^{-1} \mid a, b \in G\}$ is called the <u>commutant</u> of the group G, labelled $K(G)$

<u>Remark:</u> G is commutative iff $K(G) = \{e\}$

Two important facts of $K(G)$ are HW problems:

Remark: $K(G)$ is a subgp of G, and, in particular, $K(G)$ is normal. (HW) eg. Consider the n-gon

Symmetries $D_{2\cdot n}$ (flips and rotations)  & $ghg^{-1}h^{-1} \in K(D_{2n})$

Recall orientation: flips reverse, rotations preserve

Consider

	Orientation	
$R\cdot R$:	preserve	$\to$ rotation
$R\cdot F$:	reverse	$\to$ Flip
check two flips $\to$ $F\cdot F$:	preserve	$\to$ rotation

Also the inverse of a rotation is a rot. & of a flip is a flip. Hence

$g\cdot h\cdot g^{-1}\cdot h^{-1}$ must preserve orientation!

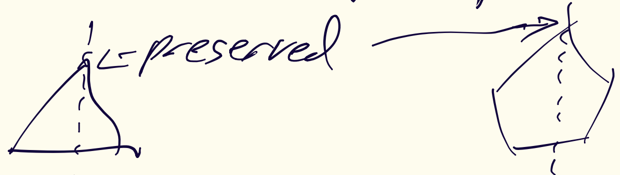
eg. $F\cdot R\cdot F^{-1}\cdot R^{-1}$

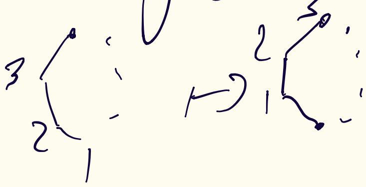
Reverse — preserve

Cartesian product ~~deja~~ vs direct

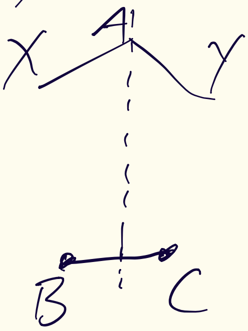
So only rotations appear in $K(D_{2n})$

If n is odd then flips preserve one vertex:



Can a single rotation, i.e. a rotation by one turn  occur in $K(D_{2n})$?

If so then, as it is a group, every power of this rotation would also occur i.e. $K(D_{2n}) = \mathbb{Z}_n$

Check: 

let F flip through this axis & let R be the ccw rot sending A to B.

Then $FRF^{-1}R^{-1}$ is a rotation sending:

$R^{-1}: B \mapsto A$

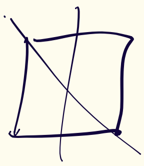
$F^{-1}: A \mapsto A$

$R: A \mapsto B$

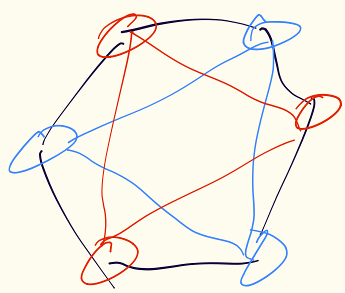
$F: B \mapsto C$

So $FRF^{-1}R^{-1}$ is a rot. sending B to C. This elt generates all rotations, hence

$K(D_{2n}) \cong \mathbb{Z}_n$ (Note $RF^{-1}R^{-1}$ preserved B, so is the flip through B)

What if n is even? Then flips preserve 2 vertices 

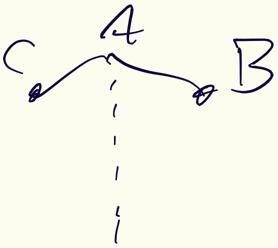
Consider the sequences of even & odd vertices seperately.



One can think of two inscribed polygons defined by these sequences.
If $g \in D_{2n}$ then g swaps the two polygons or preserves them. Whichever happens, g^{-1} does the same.

Hence any $ghg^{-1}h^{-1}$ must preserve the polygons! eg. swap · preserve · swap · preserve = preserve.
In other words, rotations by a single turn $(1\,2\,3\ldots n)$ can't appear in $K(D_{2n})$!

What about $(1\,3\,5\ldots n{-}1\;2\,4\,6\ldots n{-}2)$ (2 turns)?

Consider 

F flip through A axis
R Rotation $B \mapsto A$, $A \mapsto C$ etc

Then $RFR^{-1}F^{-1}$: Rotation sending B to C.

$F^{-1}: B \mapsto C$
$R^{-1}: C \mapsto A$
$F: A \mapsto A$
$R: A \mapsto C$

Hence $D_{2n} \cong \mathbb{Z}_{n/2}$ when n even

$\boxed{\text{Eg}}$ Consider $K(G) \trianglelefteq G$. Then we can consider the quotient $G/K(G)$

Note that $gh = hg$ iff $ghg^{-1}h^{-1} = e$

Hence, the non trivial elements of $K(G)$ come from non-commuting elements of G... So we expect $G/K(G)$ to be commutative!

Let g & $h \in G/K(G)$ i.e. $gK(G)$ & $hK(G)$

Now $(g \cdot h)K(G) = (h \cdot g)K(G)$ iff $(ghg^{-1}h^{-1})K(G)$

$= e\,K(G) = K(G)$

But this is precisely when $ghg^{-1}h^{-1} \in K(G)$ which follows by def^n.

So $G/K(G)$ is abelian $\forall$ groups G.